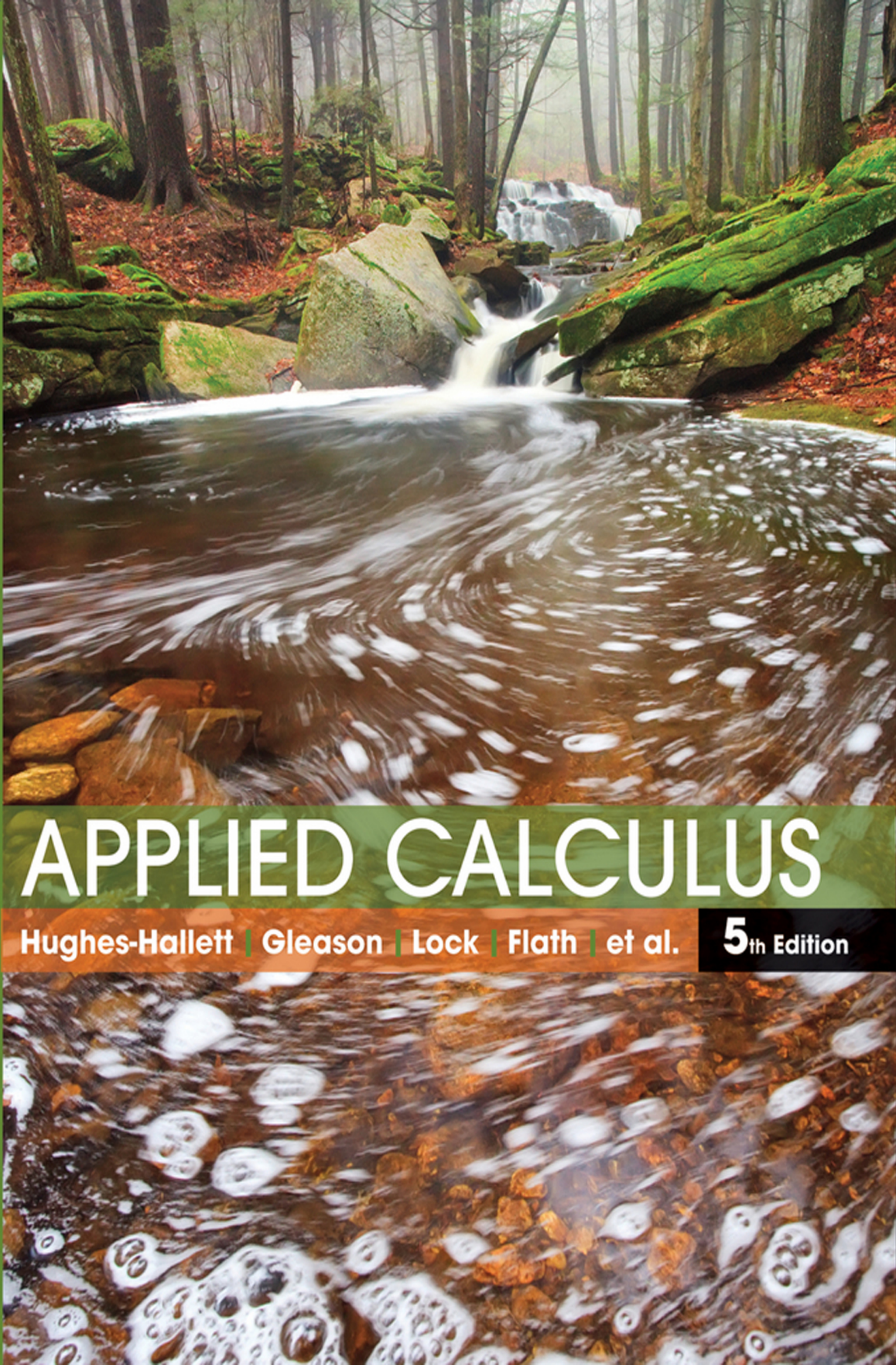

APPLIED CALCULUS

Hughes-Hallett | Gleason | Lock | Flath | et al.

5th Edition

APPLIED CALCULUS

Fifth Edition

Produced by the Calculus Consortium and initially funded by a National Science Foundation Grant.

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We dedicate this book to Andrew M. Gleason.

His brilliance and the extraordinary kindness and dignity with which he treated others made an enormous difference to us, and to many, many people. Andy brought out the best in everyone.

Deb Hughes Hallett
for the Calculus Consortium

PREFACE

Calculus is one of the greatest achievements of the human intellect. Inspired by problems in astronomy, Newton and Leibniz developed the ideas of calculus 300 years ago. Since then, each century has demonstrated the power of calculus to illuminate questions in mathematics, the physical sciences, engineering, and the social and biological sciences.

Calculus has been so successful because of its extraordinary power to reduce complicated problems to simple rules and procedures. Therein lies the danger in teaching calculus: it is possible to teach the subject as nothing but the rules and procedures—thereby losing sight of both the mathematics and of its practical value. This edition of *Applied Calculus* continues our effort to promote courses in which understanding reinforces computation.

Mathematical Concepts and Modeling

The first stage in the development of mathematical thinking is the acquisition of a clear intuitive picture of the central ideas. In the next stage, the student learns to reason with the intuitive ideas in plain English. After this foundation has been laid, there is a choice of direction. All students benefit from both mathematical concepts and modeling, but the balance may differ for different groups of students. For instructors wishing to emphasize the connection between calculus and other fields, the text includes:

- A variety of problems and examples from the **biological sciences** and **economics**.
- Models from the **health sciences** and of **population growth**.
- New problems on **sustainability**.
- New case studies on **medicine** by David E. Sloane, MD.

Active Learning: Good Problems

As instructors ourselves, we know that interactive classrooms and well-crafted problems promote student learning. Since its inception, the hallmark of our text has been its innovative and engaging problems. These problems probe student understanding in ways often taken for granted. Praised for their creativity and variety, the influence of these problems has extended far beyond the users of our textbook.

The Fifth Edition continues this tradition. Under our approach, which we called the "Rule of Four," ideas are presented graphically, numerically, symbolically, and verbally, thereby encouraging students with a variety of learning styles to expand their knowledge. This edition expands the types of problems available:

- End of **Section Problems** reinforce the ideas of that section and make connections with earlier sections; **Chapter Review Problems** ask students to review ideas from the whole chapter.
- **ConcepTests** promote active learning in the classroom. These can be used with or without clickers (personal response systems), and have been shown to dramatically improve student learning. ConcepTests are particularly useful to instructors teaching in a "flipped classroom." Available in a book or on the web at www.wiley.com/college/hughes-hallett.
- **Projects** at the end of each chapter provide opportunities for a sustained investigation, often using skills from different parts of the course. These now include medical case studies based on clinical practice.
- True-False **Strengthen Your Understanding** questions at the end of every chapter enable students to check their progress.
- **Spreadsheet Projects** in the Appendix to the book provide the opportunity for students to develop their spreadsheet skills while deepening their understanding of functions and calculus.
- **Focus on Practice** exercises at the end of Chapter 3 and 6 (Differentives and Antiderivatives) build student skill and confidence.

- **Online Problems** available in WileyPLUS or WeBWorK, for example. Many problems are randomized, providing students with expanded opportunities for practice with immediate feedback.

Origin of the Text: A Community of Instructors

This text, like others we write, draws on the experience of a diverse group of authors and users. We have benefitted enormously from input from a broad spectrum of instructors—at research universities, four-year colleges, community colleges, and secondary schools. For *Applied Calculus*, the contributions of colleagues in biology, economics, medicine, business, and other life and social sciences have been equally central to the development of the text. It is the collective wisdom of this community of mathematicians, teachers, natural and social scientists that forms the basis for the new edition.

What Student Background is Expected?

This book is intended for students in business, the social sciences, and the life sciences. A background in trigonometry is *not* required; the sections involving trigonometry are optional.

We have found the material to be thought-provoking for well-prepared students while still accessible to students with limited algebra backgrounds. Providing numerical and graphical approaches as well as the algebraic gives students several ways of mastering the material. This approach encourages students to persist, thereby lowering failure rate; a pre-test over background material is available in the appendix to the book; An algebra refresher is avalable at the student book companion site at www.wiley.com/college/hughes-hallett.

Mathematical Skills: A Balance Between Symbolic Manipulation and Technology

To use calculus effectively, students need familiarity with both symbolic manipulation and the use of technology. The balance between them may vary, depending on the needs of the students and the wishes of the instructor. The book is adaptable to many different combinations.

The book does not require any specific software or technology. It has been used with graphing calculators, many types of software, including computer algebra systems. Any technology with the ability to graph functions and perform numerical integration will suffice. Students are expected to use their own judgment to determine where technology is useful.

The Fifth Edition

Because different users often choose very different topics to cover in a one-semester applied calculus course, we have designed this book for either a one-semester course (with much flexibility in choosing topics) or a two-semester course. Sample syllabi are provided in the Instructor's Manual.

The fifth edition has the same vision as previous editions. In preparing this edition, we solicited comments from a large number of mathematics instructors who had used the text. We continued to discuss with our colleagues in client disciplines the mathematical needs of their students. We were offered many valuable suggestions, which we have tried to incorporate, while maintaining our original commitment to a focused treatment of a limited number of topics. The changes we have made include:

- **Updated data** and **fresh applications** throughout the book, including
  - New problems on **sustainability**.
  - New case studies on **medicine** by David E. Sloane, MD.
- Many **new problems** have been added, designed to build student confidence with basic concepts and to reinforce skills.
- The material on **integration** has been **streamlined and reorganized**.
  - In **Chapter 5**, Sections 5.1-5.5 have been streamlined.
  - **Section 5.6** on Average Value is the former Section 6.1.
  - **Chapters 6,7** have been rearranged and combined, putting an introduction to antiderivatives before the applications to consumer surplus and present value. This gives instructors the choice of evaluating definite integrals numerically or using the Fundamental Theorem of Calculus.

- New **projects** have been added in Chapters 1, 2, 3, 4, 5, and 10.
- As in the previous edition, a **Pre-test** is included for students whose skills may need a refresher prior to taking the course.

Content

This content represents our vision of how applied calculus can be taught. It is flexible enough to accommodate individual course needs and requirements. Topics can easily be added or deleted, or the order changed.

Chapter 1: Functions and Change

Chapter 1 introduces the concept of a function and the idea of change, including the distinction between total change, rate of change, and relative change. All elementary functions are introduced here. Although the functions are probably familiar, the graphical, numerical, verbal, and modeling approach to them is likely to be new. We introduce exponential functions early, since they are fundamental to the understanding of real-world processes. The trigonometric functions are optional.

A brief introduction to elasticity has been added to Section 1.3.

Chapter 2: Rate of Change: The Derivative

Chapter 2 presents the key concept of the derivative according to the Rule of Four. The purpose of this chapter is to give the student a practical understanding of the meaning of the derivative and its interpretation as an instantaneous rate of change. Students will learn how the derivative can be used to represent relative rates of change. After finishing this chapter, a student will be able to approximate derivatives numerically by taking difference quotients, visualize derivatives graphically as the slope of the graph, and interpret the meaning of first and second derivatives in various applications. The student will also understand the concept of marginality and recognize the derivative as a function in its own right.

Focus on Theory: This section discusses limits and continuity and presents the symbolic definition of the derivative.

Chapter 3: Short-Cuts to Differentiation

The derivatives of all the functions in Chapter 1 are introduced, as well as the rules for differentiating products, quotients, and composite functions. Students learn how to find relative rates of change using logarithms.

Focus on Theory: This section uses the definition of the derivative to obtain the differentiation rules.

Focus on Practice: This section provides a collection of differentiation problems for skill-building.

Chapter 4: Using the Derivative

The aim of this chapter is to enable the student to use the derivative in solving problems, including optimization and graphing. It is not necessary to cover all the sections.

Chapter 5: Accumulated Change: The Definite Integral

Chapter 5 presents the key concept of the definite integral, in the same spirit as Chapter 2.

The purpose of this chapter is to give the student a practical understanding of the definite integral as a limit of Riemann sums, and to bring out the connection between the derivative and the definite integral in the Fundamental Theorem of Calculus. We use the same method as in Chapter 2, introducing the fundamental concept in depth without going into technique. The student will finish the chapter with a good grasp of the definite integral as a limit of Riemann sums, and the ability to approximate a definite integral numerically and interpret it graphically. The chapter includes applications of definite integrals in a variety of contexts, including the average value of a function.

Chapter 5 can be covered immediately after Chapter 2 without difficulty.

The introduction to the definite integral has been streamlined. Average values, formerly in Section 6.1, are now in Section 5.6.

Focus on Theory: This section presents the Second Fundamental Theorem of Calculus and the properties of the definite integral.

Chapter 6: Antiderivatives and Applications

This chapter combines the former Chapter 6 and 7. It covers antiderivatives from a graphical, numerical, and algebraic point of view. The Fundamental Theorem of Calculus is used to evaluate definite integrals.

Optional application sections are included on consumer and producer surplus and on present and future value; the integrals in these sections can be evaluated numerically or using the Fundamental Theorem. The chapter concludes with optional sctions on integration by substitution and integration by parts.

Section 6.1, on graphical and numerical antiderivatives, is based on the former Section 7.5. Section 6.2, on symbolic antiderivatives, is based on the former Section 7.1. Using the Fundamental Theorem to find definite integrals is in Section 6.3, formerly Section 7.3. Sections 6.4 and 6.5 are the former Sections 6.2 and 6.3. Sections 6.6 and 6.7 are the former Sections 7.2 and 7.4.

Focus on Practice: This section provides a collection of integration problems for skill-building.

Chapter 7: Probability

This chapter covers probability density functions, cumulative distribution functions, the median and the mean.

Chapter 7 is the former Chapter 8.

Chapter 8: Functions of Several Variables

This chapter introduces functions of two variables from several points of view, using contour diagrams, formulas, and tables. It gives students the skills to read contour diagrams and think graphically, to read tables and think numerically, and to apply these skills, along with their algebraic skills, to modeling. The idea of the partial derivative is introduced from graphical, numerical, and symbolic viewpoints. Partial derivatives are then applied to optimization problems, ending with a discussion of constrained optimization using Lagrange multipliers.

Chapter 8 is the former Chapter 9.

Focus on Theory: This section uses optimization to derive the formula for the regression line.

Chapter 9: Mathematical Modeling Using Differential Equations

This chapter introduces differential equations. The emphasis is on modeling, qualitative solutions, and interpretation. This chapter includes applications of systems of differential equations to population models, the spread of disease, and predator-prey interactions.

Chapter 9 is the former Chapter 10.

Focus on Theory: This section explains the technique of separation of variables.

Chapter 10: Geometric Series

This chapter covers geometric series and their applications to business, economics, and the life sciences.

Chpater 10 is the former Chapter 11.

Appendices

The first appendix introduces the student to fitting formulas to data; the second appendix provides further discussion of compound interest and the definition of the number e. The third appendix contains a selection of spreadsheet projects.

Supplementary Materials

Supplements for the instructor can be obtained by sending a request on your institutional letterhead to Mathematics Marketing Manager, John Wiley & Sons Inc., 111 River Street, Hoboken, NJ 07030-5774, or by contacting your local Wiley representative. The following supplementary materials are available.

- **Instructor's Manual** (ISBN 978-1-118-71506-2) containing teaching tips, sample syllabii, calculator programs, and overhead transparency masters.

- **Instructor's Solution Manual** (ISBN 978-1-118-71498-0) with complete solutions to all problems.
- **Student's Solution Manual** (ISBN 978-1-118-71499-7) with complete solutions to half the odd-numbered problems.
- **Additional Material for Instructors**, elaborating specially marked points in the text, as well as password protected electronic versions of the instructor ancillaries, can be found on the web at www.wiley.com/college/hughes-hallett.
- **Additional Material for Students**, at the student book companion site at www.wiley.com/college/hughes-hallett, includes an algebra refresher and web quizzes.

Getting Started Technology Manual Series:

- **Getting Started with Mathematica**, 3rd edn, by C-K. Cheung, G.E. Keough, Robert H. Gross, and Charles Landraitis of Boston College (ISBN 978-0-470-45687-3)
- **Getting Started with Maple**, 3rd edn, by C-K. Cheung, G.E. Keough, both of Boston College, and Michael May of St. Louis University (ISBN 978-0-470-45554-8)

ConcepTests

ConcepTests (ISBN 978-1-118-71494-2), or clicker questions, modeled on the pioneering work of Harvard physicist Eric Mazur, are questions designed to promote active learning during class, particularly (but not exclusively) in large lectures. Evaluation data shows that students taught with ConcepTests outperformed students taught by traditional lecture methods 73% versus 17% on conceptual questions, and 63% versus 54% on computational problems.[1] A supplement to *Applied Calculus*, 5th edn, containing ConcepTests by section, is available from your Wiley representative.

Wiley Faculty Network

The Wiley Faculty Network is a peer-to-peer network of academic faculty dedicated to the effective use of technology in the classroom. This group can help you apply innovative classroom techniques and implement specific software packages. Visit www.wherefacultyconnect.com or ask your Wiley representative for details.

WileyPLUS

WileyPLUS, Wiley's digital learning environment, is loaded with all of the supplements above, and also features:

- E-book, which is an exact version of the print text, but also features hyperlinks to questions, definitions, and supplements for quicker and easier support.
- Homework management tools, which easily enable the instructor to assign and automatically grade questions, using a rich set of options and controls.
- QuickStart pre-designed reading and homework assignments. Use them as-is or customize them to fit the needs of your classroom.
- Guided Online (GO) Exercises, which prompt students to build solutions step-by-step. Rather than simply grading an exercise answer as wrong, GO problems show students precisely where they are making a mistake.
- Algebra & Trigonometry Refresher quizzes, which provide students with an opportunity to brush-up on material necessary to master calculus, as well as to determine areas that require further review.
- Graphing Calculator Manual, to help students get the most out of their graphing calculator, and to show how they can apply the numerical and graphing functions of their calculators to their study of calculus.

[1] "Peer Instruction in Physics and Mathematics" by Scott Pilzer in *Primus*, Vol XI, No 2, June 2001. At the start of Calculus II, students earned 73% on conceptual questions and 63% on computational questions if they were taught with ConcepTests in Calculus I; 17% and 54% otherwise.

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Jeff Tecosky-Feldman
Thomas W. Tucker
Aaron D. Wooton

APPLICATIONS INDEX

Business and Economics

Life Sciences and Ecology

Physical Sciences

Social Sciences

To Students: How to Learn from this Book

- This book may be different from other math textbooks that you have used, so it may be helpful to know about some of the differences in advance. At every stage, this book emphasizes the *meaning* (in practical, graphical or numerical terms) of the symbols you are using. There is much less emphasis on "plug-and-chug" and using formulas, and much more emphasis on the interpretation of these formulas than you may expect. You will often be asked to explain your ideas in words or to explain an answer using graphs.
- The book contains the main ideas of calculus in plain English. Success in using this book will depend on reading, questioning, and thinking hard about the ideas presented. It will be helpful to read the text in detail, not just the worked examples.
- There are few examples in the text that are exactly like the homework problems, so homework problems can't be done by searching for similar–looking "worked out" examples. Success with the homework will come by grappling with the ideas of calculus.
- For many problems in the book, there is more than one correct approach and more than one correct solution. Sometimes, solving a problem relies on common sense ideas that are not stated in the problem explicitly but which you know from everyday life.
- Some problems in this book assume that you have access to a graphing calculator or computer. There are many situations where you may not be able to find an exact solution to a problem, but you can use a calculator or computer to get a reasonable approximation.
- This book attempts to give equal weight to four methods for describing functions: graphical (a picture), numerical (a table of values), algebraic (a formula), and verbal (words). Sometimes it's easier to translate a problem given in one form into another. For example, you might replace the graph of a parabola with its equation, or plot a table of values to see its behavior. It is important to be flexible about your approach: if one way of looking at a problem doesn't work, try another.
- Students using this book have found discussing these problems in small groups helpful. There are a great many problems which are not cut-and-dried; it can help to attack them with the other perspectives your colleagues can provide. If group work is not feasible, see if your instructor can organize a discussion session in which additional problems can be worked on.
- You are probably wondering what you'll get from the book. The answer is, if you put in a solid effort, you will get a real understanding of one of the crowning achievements of human creativity—calculus—as well as a real sense of the power of mathematics in the age of technology.

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Chapter One

FUNCTIONS AND CHANGE

Contents

1.1 WHAT IS A FUNCTION?

In mathematics, a *function* is used to represent the dependence of one quantity upon another.

Let's look at an example. Syracuse, New York has the highest annual snowfall of any US city because of the "lake-effect" snow coming from cold Northwest winds blowing over nearby Lake Ontario. Lake-effect snowfall has been heavier over the last few decades; some have suggested this is due to the warming of Lake Ontario by climate change. In December 2010, Syracuse got 66.9 inches of snow in one 12-day period, all of it from lake-effect snow. See Table 1.1.

Table 1.1 *Daily snowfall in Syracuse, December 5–16, 2010*

Date (December 2010)	5	6	7	8	9	10	11	12	13	14	15	16
Snowfall in inches	6.8	12.2	9.3	14.9	1.9	0.1	0.0	0.0	1.4	5.0	11.9	3.4

You may not have thought of something so unpredictable as daily snowfall as being a function, but it *is* a function of date, because each day gives rise to one snowfall total. There is no formula for the daily snowfall (otherwise we would not need a weather bureau), but nevertheless the daily snowfall in Syracuse does satisfy the definition of a function: Each date, t, has a unique snowfall, S, associated with it.

We define a function as follows:

> A **function** is a rule that takes certain numbers as inputs and assigns to each a definite output number. The set of all input numbers is called the **domain** of the function and the set of resulting output numbers is called the **range** of the function.

The input is called the *independent variable* and the output is called the *dependent variable*. In the snowfall example, the domain is the set of December dates $\{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\}$ and the range is the set of daily snowfalls $\{0.0, 0.1, 1.4, 1.9, 3.4, 5.0, 6.8, 9.3, 11.9, 12.2, 14.9\}$. We call the function f and write $S = f(t)$. Notice that a function may have identical outputs for different inputs (December 11 and 12, for example).

Some quantities, such as date, are *discrete*, meaning they take only certain isolated values (dates must be integers). Other quantities, such as time, are *continuous* as they can be any number. For a continuous variable, domains and ranges are often written using interval notation:

The set of numbers t such that $a \le t \le b$ is called a *closed interval* and written $[a, b]$.

The set of numbers t such that $a < t < b$ is called an *open interval* and written (a, b).

The Rule of Four: Tables, Graphs, Formulas, and Words

Functions can be represented by tables, graphs, formulas, and descriptions in words. For example, the function giving the daily snowfall in Syracuse can be represented by the graph in Figure 1.1, as well as by Table 1.1.

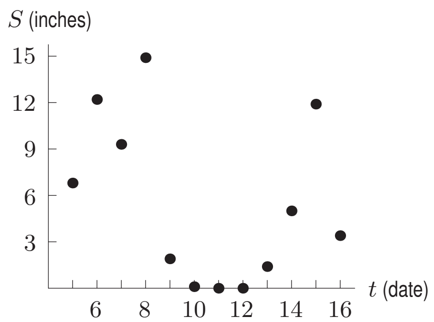


Figure 1.1: Syracuse snowfall, December, 2010

Other functions arise naturally as graphs. Figure 1.2 contains electrocardiogram (EKG) pictures showing the heartbeat patterns of two patients, one normal and one not. Although it is possible to construct a formula to approximate an EKG function, this is seldom done. The pattern of repetitions is what a doctor needs to know, and these are more easily seen from a graph than from a formula. However, each EKG gives electrical activity as a function of time.

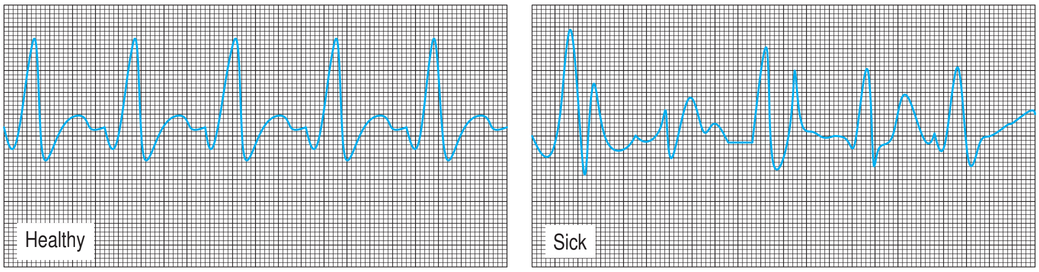


Figure 1.2: EKG readings on two patients

As another example of a function, consider the snow tree cricket. Surprisingly enough, all such crickets chirp at essentially the same rate if they are at the same temperature. That means that the chirp rate is a function of temperature. In other words, if we know the temperature, we can determine the chirp rate. Even more surprisingly, the chirp rate, C, in chirps per minute, increases steadily with the temperature, T, in degrees Fahrenheit, and can be computed, to a fair degree of accuracy, using the formula

$$C = f(T) = 4T - 160.$$

Mathematical Modeling

A *mathematical model* is a mathematical description of a real situation. In this book we consider models that are functions, such as $C = f(T) = 4T - 160$.

Modeling almost always involves some simplification of reality. We choose which variables to include and which to ignore—for example, we consider the dependence of chirp rate on temperature, but not on other variables. The choice of variables is based on knowledge of the context (the biology of crickets, for example), not on mathematics. To test the model, we compare its predictions with observations.

In this book, we often model a situation that has a discrete domain with a continuous function whose domain is an interval of numbers. For example, the annual US gross domestic product (GDP) has a value for each year, $t = 0, 1, 2, 3, \ldots$. We may model it by a function of the form $G = f(t)$, with values for t in a continuous interval. In doing this, we expect that the values of $f(t)$ match the values of the GDP at the points $t = 0, 1, 2, 3, \ldots$, and that information obtained from $f(t)$ closely matches observed values.

Used judiciously, a mathematical model captures trends in the data to enable us to analyze and make predictions. A common way of finding a model is described in Appendix A.

Function Notation and Intercepts

We write $y = f(t)$ to express the fact that y is a function of t. The independent variable is t, the dependent variable is y, and f is the name of the function. The graph of a function has an *intercept* where it crosses the horizontal or vertical axis. Horizontal intercepts are also called the *zeros* of the function.

Example 1 (a) Graph the cricket chirp rate function, $C = f(T) = 4T - 160$.
(b) Solve $f(T) = 0$ and interpret the result.

Solution (a) The graph is in Figure 1.3.

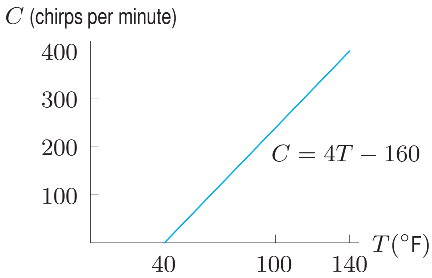


Figure 1.3: Cricket chirp rate as a function of temperature

(b) Solving $f(T) = 0$ gives the horizontal intercept:

$$4T - 160 = 0$$
$$T = \frac{160}{4} = 40.$$

Thus at a temperature of $40°$F, the chirp rate is zero.

For temperatures below $40°$F, the model would predict negative values of C, so we conclude that the model does not apply for such temperature values.

Example 2 The value of a car, V, is a function of the age of the car, a, so $V = g(a)$, where g is the name we are giving to this function.

(a) Interpret the statement $g(5) = 9$ in terms of the value of a car if V is in thousands of dollars and a is in years.

(b) In the same units, the value of a Honda[1] is approximated by $g(a) = 13.78 - 0.8a$. Find and interpret the vertical and horizontal intercepts of the graph of this depreciation function g.

Solution

(a) Since $V = g(a)$, the statement $g(5) = 9$ means $V = 9$ when $a = 5$. This tells us that the car is worth \$9000 when it is 5 years old.

(b) Since $V = g(a)$, a graph of the function g has the value of the car on the vertical axis and the age of the car on the horizontal axis. The vertical intercept is the value of V when $a = 0$. It is $V = g(0) = 13.78$, so the Honda was valued at \$13,780 when new. The horizontal intercept is the value of a such that $g(a) = 0$, so

$$13.78 - 0.8a = 0$$
$$a = \frac{13.78}{0.8} = 17.2.$$

At age 17 years, the Honda has no value.

Increasing and Decreasing Functions

In the previous examples, the chirp rate increases with temperature, while the value of the Honda decreases with age. We express these facts saying that f is an increasing function, while g is decreasing. See Figure 1.4. In general:

> A function f is **increasing** if the values of $f(x)$ increase as x increases.
> A function f is **decreasing** if the values of $f(x)$ decrease as x increases.
>
> The graph of an *increasing* function *climbs* as we move from left to right.
> The graph of a *decreasing* function *descends* as we move from left to right.

[1] From data obtained from the Kelley Blue Book, www.kbb.com.

Figure 1.4: Increasing and decreasing functions

Problems for Section 1.1

1. Which graph in Figure 1.5 best matches each of the following stories?[2] Write a story for the remaining graph.
   (a) I had just left home when I realized I had forgotten my books, so I went back to pick them up.
   (b) Things went fine until I had a flat tire.
   (c) I started out calmly but sped up when I realized I was going to be late.

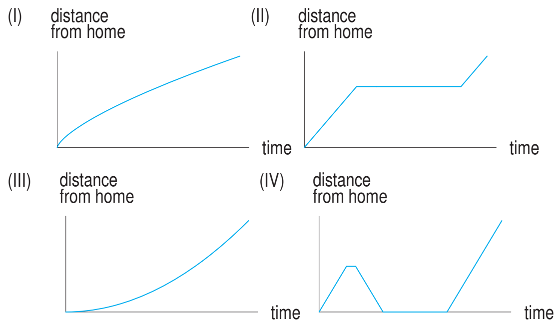


Figure 1.5

In Problems 2–5, use the description of the function to sketch a possible graph. Put a label on each axis and state whether the function is increasing or decreasing.

2. The height of a sand dune is a function of time, and the wind erodes away the sand dune over time.
3. The amount of carbon dioxide in the atmosphere is a function of time, and is going up over time.
4. The number of air conditioning units sold is a function of temperature, and goes up as the temperature goes up.
5. The noise level, in decibels, is a function of distance from the source of the noise, and the noise level goes down as the distance increases.
6. The population of Washington DC grew from 1900 to 1950, stayed approximately constant during the 1950s, and decreased from about 1960 to 2005. Graph the population as a function of years since 1900.
7. Let $W = f(t)$ represent wheat production in Argentina,[3] in millions of metric tons, where t is years since 2006. Interpret the statement $f(4) = 14$ in terms of wheat production.
8. The concentration of carbon dioxide, $C = f(t)$, in the atmosphere, in parts per million (ppm), is a function of years, t, since 1960.
   (a) Interpret $f(40) = 370$ in terms of carbon dioxide.[4]
   (b) What is the meaning of $f(50)$?
9. (a) The graph of $r = f(p)$ is in Figure 1.6. What is the value of r when p is 0? When p is 3?
   (b) What is $f(2)$?

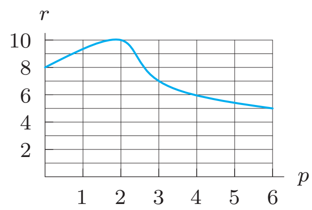
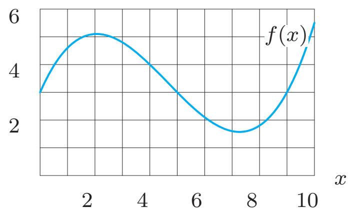


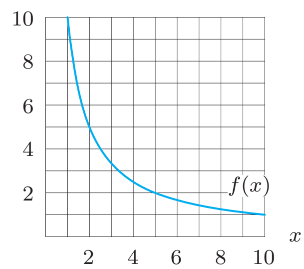
Figure 1.6

For the functions in Problems 10–14, find $f(5)$.

10. $f(x) = 2x + 3$
11. $f(x) = 10x - x^2$
12.



13.



14.

x	1	2	3	4	5	6	7	8
$f(x)$	2.3	2.8	3.2	3.7	4.1	5.0	5.6	6.2

[2] Adapted from Jan Terwel, "Real Math in Cooperative Groups in Secondary Education." *Cooperative Learning in Mathematics*, ed. Neal Davidson, p. 234 (Reading: Addison Wesley, 1990).
[3] http://ageconsearch.umn.edu/bitstream/115558/2/AAE680.pdf, accessed September 2012.
[4] *Vital Signs 2007-2008*, The Worldwatch Institute, W.W. Norton & Company, 2007, p. 43.

15. Let $y = f(x) = x^2 + 2$.

(a) Find the value of y when x is zero.
(b) What is $f(3)$?
(c) What values of x give y a value of 11?
(d) Are there any values of x that give y a value of 1?

In Problems 16–19 the function $S = f(t)$ gives the average annual sea level, S, in meters, in Aberdeen, Scotland,[5] as a function of t, the number of years before 2008. Write a mathematical expression that represents the given statement.

16. In 1983 the average annual sea level in Aberdeen was 7.019 meters.

17. The average annual sea level in Aberdeen in 2008.

18. The average annual sea level in Aberdeen was the same in 1865 and 1911.

19. The average annual sea level in Aberdeen increased by 1 millimeter from 2007 to 2008.

20. **(a)** A potato is put in an oven to bake at time $t = 0$. Which of the graphs in Figure 1.7 could represent the potato's temperature as a function of time?
(b) What does the vertical intercept represent in terms of the potato's temperature?

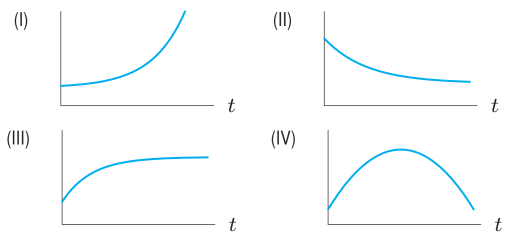


Figure 1.7

21. An object is put outside on a cold day at time $t = 0$. Its temperature, $H = f(t)$, in °C, is graphed in Figure 1.8.

(a) What does the statement $f(30) = 10$ mean in terms of temperature? Include units for 30 and for 10 in your answer.
(b) Explain what the vertical intercept, a, and the horizontal intercept, b, represent in terms of temperature of the object and time outside.

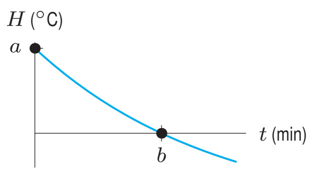


Figure 1.8

22. In the Andes mountains in Peru, the number, N, of species of bats is a function of the elevation, h, in feet above sea level, so $N = f(h)$.

(a) Interpret the statement $f(500) = 100$ in terms of bat species.
(b) What are the meanings of the vertical intercept, k, and horizontal intercept, c, in Figure 1.9?

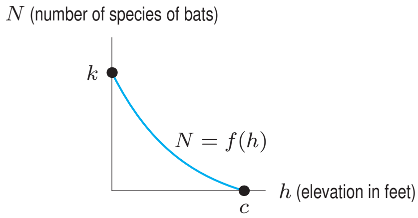


Figure 1.9

23. In tide pools on the New England coast, snails eat algae. Describe what Figure 1.10 tells you about the effect of snails on the diversity of algae.[6] Does the graph support the statement that diversity peaks at intermediate predation levels?

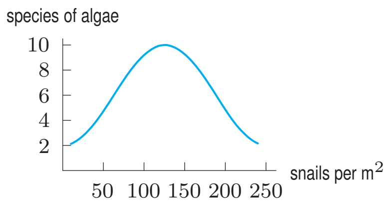


Figure 1.10

24. Figure 1.11 shows the amount of nicotine, $N = f(t)$, in mg, in a person's bloodstream as a function of the time, t, in hours, since the person finished smoking a cigarette.

(a) Estimate $f(3)$ and interpret it in terms of nicotine.
(b) About how many hours have passed before the nicotine level is down to 0.1 mg?
(c) What is the vertical intercept? What does it represent in terms of nicotine?
(d) If this function had a horizontal intercept, what would it represent?

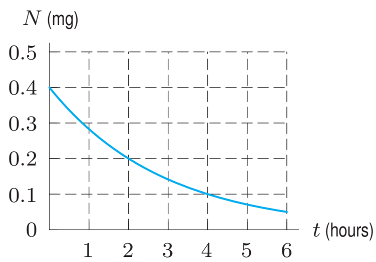


Figure 1.11

[5] www.decc.gov.uk, accessed June 2011
[6] Rosenzweig, M.L., *Species Diversity in Space and Time,* p. 343 (Cambridge: Cambridge University Press, 1995).

25. A deposit is made into an interest-bearing account. Figure 1.12 shows the balance, B, in the account t years later.

(a) What was the original deposit?
(b) Estimate $f(10)$ and interpret it.
(c) When does the balance reach \$5000?

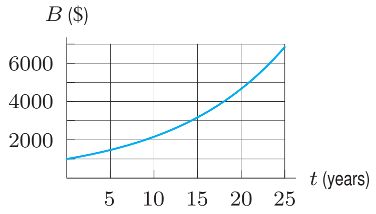


Figure 1.12

26. The use of CFCs (chlorofluorocarbons) has declined since the 1987 Montreal Protocol came into force to reduce the use of substances that deplete the ozone layer. World annual CFC consumption, $C = f(t)$, in million tons, is a function of time, t, in years since 1987. (CFCs are measured by the weight of ozone that they could destroy.)

(a) Interpret $f(10) = 0.2$ in terms of CFCs.[7]
(b) Interpret the vertical intercept of the graph of this function in terms of CFCs.
(c) Interpret the horizontal intercept of the graph of this function in terms of CFCs.

27. When a patient with a rapid heart rate takes a drug, the heart rate plunges dramatically and then slowly rises again as the drug wears off. Sketch the heart rate against time from the moment the drug is administered.

28. The gas mileage of a car (in miles per gallon) is highest when the car is going about 45 miles per hour and is lower when the car is going faster or slower than 45 mph. Graph gas mileage as a function of speed of the car.

29. After an injection, the concentration of a drug in a patient's body increases rapidly to a peak and then slowly decreases. Graph the concentration of the drug in the body as a function of the time since the injection was given. Assume that the patient has none of the drug in the body before the injection. Label the peak concentration and the time it takes to reach that concentration.

30. Financial investors know that, in general, the higher the expected rate of return on an investment, the higher the corresponding risk.

(a) Graph this relationship, showing expected return as a function of risk.
(b) On the figure from part (a), mark a point with high expected return and low risk. (Investors hope to find such opportunities.)

31. The number of sales per month, S, is a function of the amount, a (in dollars), spent on advertising that month, so $S = f(a)$.

(a) Interpret the statement $f(1000) = 3500$.
(b) Which of the graphs in Figure 1.13 is more likely to represent this function?
(c) What does the vertical intercept of the graph of this function represent, in terms of sales and advertising?

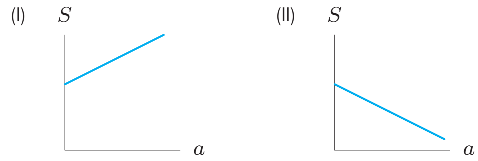


Figure 1.13

32. Figure 1.14 shows fifty years of fertilizer use in the US, India, and the former Soviet Union.[8]

(a) Estimate fertilizer use in 1970 in the US, India, and the former Soviet Union.
(b) Write a sentence for each of the three graphs describing how fertilizer use has changed in each region over this 50-year period.

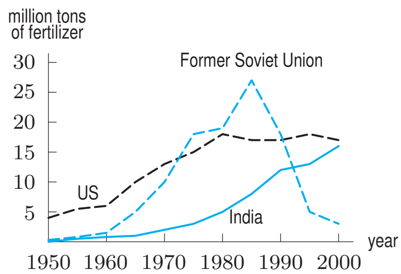


Figure 1.14

33. The six graphs in Figure 1.15 show frequently observed patterns of age-specific cancer incidence rates, in number of cases per 1000 people, as a function of age.[9] The scales on the vertical axes are equal.

(a) For each of the six graphs, write a sentence explaining the effect of age on the cancer rate.
(b) Which graph shows a relatively high incidence rate for children? Suggest a type of cancer that behaves this way.
(c) Which graph shows a brief decrease in the incidence rate at around age 50? Suggest a type of cancer that might behave this way.

[7] *Vital Signs 2007-2008*, The Worldwatch Institute, W.W. Norton & Company, 2007, p. 47.
[8] The Worldwatch Institute, *Vital Signs 2001,* p. 32 (New York: W.W. Norton, 2001).
[9] Abraham M. Lilienfeld, *Foundations of Epidemiology*, p. 155 (New York: Oxford University Press, 1976).

(d) Which graph or graphs might represent a cancer that is caused by toxins which build up in the body over time? (For example, lung cancer.) Explain.

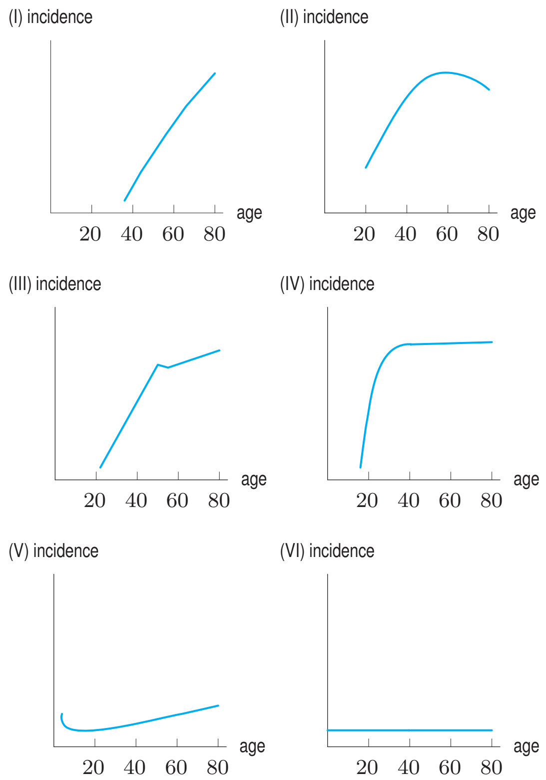


Figure 1.15

34. Table 1.2 shows the average annual sea level, S, in meters, in Aberdeen, Scotland,[10] as a function of time, t, measured in years before 2008.

Table 1.2

t	0	25	50	75	100	125
S	7.094	7.019	6.992	6.965	6.938	6.957

(a) What was the average sea level in Aberdeen in 2008?

(b) In what year was the average sea level 7.019 meters? 6.957 meters?

(c) Table 1.3 gives the average sea level, S, in Aberdeen as a function of the year, x. Complete the missing values.

Table 1.3

x	1883	?	1933	1958	1983	2008
S	?	6.938	?	6.992	?	?

Problems 35–38 ask you to plot graphs based on the following story: "As I drove down the highway this morning, at first traffic was fast and uncongested, then it crept nearly bumper-to-bumper until we passed an accident, after which traffic flow went back to normal until I exited."

35. Driving speed against time on the highway

36. Distance driven against time on the highway

37. Distance from my exit vs time on the highway

38. Distance between cars vs distance driven on the highway

1.2 LINEAR FUNCTIONS

Probably the most commonly used functions are the *linear functions*, whose graphs are straight lines. The chirp-rate and the Honda depreciation functions in the previous section are both linear. We now look at more examples of linear functions.

Olympic and World Records

During the early years of the Olympics, the height of the men's winning pole vault increased approximately 8 inches every four years. Table 1.4 shows that the height started at 130 inches in 1900, and increased by the equivalent of 2 inches a year between 1900 and 1912. So the height was a linear function of time.

Table 1.4 *Winning height (approximate) for Men's Olympic pole vault*

Year	1900	1904	1908	1912
Height (inches)	130	138	146	154

If y is the winning height in inches and t is the number of years since 1900, we can write

$$y = f(t) = 130 + 2t.$$

[10]www.decc.gov.uk, accessed June 2011.